PROOF OF THE MONOTONE COLUMN
PERMANENT CONJECTURE

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In memoriam Julius Borcea

ABSTRACT. Let A be an n-by-n matrix of real numbers which are weakly decreasing down each column, $Z_n = \text{diag}(z_1, \ldots, z_n)$ a diagonal matrix of indeterminates, and J_n the n-by-n matrix of all ones. We prove that $\text{per}(J_n Z_n + A)$ is stable in the z_i, resolving a recent conjecture of Haglund and Visontai. This immediately implies that $\text{per}(zJ_n + A)$ is a polynomial in z with only real roots, an open conjecture of Haglund, Ono, and Wagner from 1999. Other applications include a multivariate stable Eulerian polynomial, a new proof of Grace's apolarity theorem and new permanental inequalities.

1. THE MONOTONE COLUMN PERMANENT CONJECTURE.

Recall that the *permanent* of an n-by-n matrix $H = (h_{ij})$ is the unsigned variant of its determinant:

$$\text{per}(H) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} h_{i,\sigma(i)},$$

with the sum over all permutations σ in the symmetric group $\mathfrak{S}_n$. A *monotone column matrix* $A = (a_{ij})$ has real entries which are weakly decreasing reading down each column: that is, $a_{ij} \geq a_{i+1,j}$ for all $1 \leq i \leq n-1$ and $1 \leq j \leq n$. Let J_n be the n-by-n matrix in which every entry is 1.

The Monotone Column Permanent Conjecture (MCPC).
If A is an n-by-n monotone column matrix then $\text{per}(zJ_n + A)$ is a polynomial in z which has only real roots.

The MCPC first appears as Conjecture 2 of [7]. (Originally with increasing columns – but the convention of decreasing columns is clearly equivalent and will be more natural later.) Theorem 1 of [7] proves the MCPC for monotone column matrices in which every entry is either 0 or 1. Other special cases appear in [6, 7, 8, 9], either for $n \leq 4$ or for rather restrictive conditions on the entries of A. In this

paper we prove the MCPC in full generality[1]. In fact we prove more. The following multivariate version of the MCPC originates in [8]. Let $Z_n = \mathrm{diag}(z_1, \ldots, z_n)$ be an n-by-n diagonal matrix of algebraically independent commuting indeterminates $\mathbf{z} = \{z_1, \ldots, z_n\}$. A polynomial $f(z_1, \ldots, z_n) \in \mathbb{C}[\mathbf{z}]$ is *stable* provided that either $f \equiv 0$ or whenever $w_j \in \mathbb{C}$ are such that $\mathrm{Im}(w_j) > 0$ for all $1 \leq j \leq n$, then $f(w_1, \ldots, w_n) \neq 0$. A stable polynomial with real coefficients is *real stable*.

The Multivariate MCP Conjecture (MMCPC).
If A is an n-by-n monotone column matrix then $\mathrm{per}(J_n Z_n + A)$ is a real stable polynomial in $\mathbb{R}[\mathbf{z}]$.

Note that $J_n Z_n + A = (z_j + a_{ij})$, so that z_j is associated with the j-th column, for each $1 \leq j \leq n$. We also write $\mathrm{per}(J_n Z_n + A) = \mathrm{per}(z_j + a_{ij})$ as it seems clearer. The MMCPC implies the MCPC since if one sets all $z_j = z$, then $\mathrm{per}(z + a_{ij})$ is a polynomial in one variable with real coefficients; this diagonalization preserves stability (see Lemma 2.1(c) below), and a univariate real polynomial is stable if and only if it has only real roots.

In Section 2 we review some results from the theory of stable polynomials which are required for our proofs. In Section 3 we reduce the MMCPC to the case of $\{0, 1\}$-matrices which are weakly decreasing down columns and weakly increasing from left to right across rows – these we call *Ferrers matrices* for convenience. Then we further transform the MMCPC for Ferrers matrices, derive a differential recurrence relation for the resulting polynomials, and use this and the results of Section 2 to prove the conjecture by induction. In Section 4 we extend these results to sub-permanents of rectangular matrices, derive a cycle-counting extension of one of them, discuss a multivariate stable generalization of Eulerian polynomials, present a new proof of Grace's apolarity theorem and derive new permanental inequalities.

2. Stable polynomials.

Over a series of several papers, Borcea and Brändén have developed the theory of stable polynomials into a powerful and flexible technique. The results we need are taken from [1, 2, 4]; see also Sections 2 and 3 of [13].

Let $\mathcal{H} = \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$ denote the open upper half of the complex plane, and let $\overline{\mathcal{H}}$ denote its closure in $\mathbb{C}$. As above, $\mathbf{z} = \{z_1, \ldots, z_n\}$ is a set of n indeterminates. For $f \in \mathbb{C}[\mathbf{z}]$ and $1 \leq j \leq n$, let $\deg_{z_j}(f)$ denote the degree of z_j in f.

Lemma 2.1 (see Lemma 2.4 of [13]). *These operations preserve stability of polynomials in $\mathbb{C}[\mathbf{z}]$.*
(a) Permutation: *for any permutation $\sigma \in \mathfrak{S}_n$, $f \mapsto f(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$.*
(b) Scaling: *for $c \in \mathbb{C}$ and $\mathbf{a} \in \mathbb{R}^n$ with $\mathbf{a} > \mathbf{0}$, $f \mapsto cf(a_1 z_1, \ldots, a_n z_n)$.*
(c) Diagonalization: *for $1 \leq i < j \leq n$, $f \mapsto f(\mathbf{z})|_{z_i = z_j}$.*
(d) Specialization: *for $a \in \overline{\mathcal{H}}$, $f \mapsto f(a, z_2, \ldots, z_n)$.*
(e) Inversion: *if $\deg_{z_1}(f) = d$, $f \mapsto z_1^d f(-z_1^{-1}, z_2, \ldots, z_n)$.*
(f) Translation: *$f \mapsto f_1 = f(z_1 + t, z_2, \ldots, z_n) \in \mathbb{C}[\mathbf{z}, t]$.*
(g) Differentiation: *$f \mapsto \partial f(\mathbf{z})/\partial z_1$.*

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[1]Thus, it was not really a permanent conjecture.

Proof. Only part (f) is not made explicit in [1, 2, 13]. But clearly if $z_1 \in \mathcal{H}$ and $t \in \mathcal{H}$ then $z_1 + t \in \mathcal{H}$, from which the result follows. $\square$

Of course, parts (d,e,f,g) apply to any index j as well, by permutation. Part (g) is the only difficult one – it is essentially the Gauss-Lucas Theorem.

Lemma 2.2. *Let $f(\mathbf{z}, t) \in \mathbb{C}[\mathbf{z}, t]$ be stable, and let*

$$f(\mathbf{z}, t) = \sum_{k=0}^{d} f_k(\mathbf{z}) t^k$$

with $f_d(\mathbf{z}) \not\equiv 0$. Then $f_k(\mathbf{z})$ is stable for all $0 \le k \le d = \deg_t(f)$.

Proof. Consider any $0 \le k \le d$. Clearly $f_k(\mathbf{z})$ is a constant multiple of $\partial^k f(\mathbf{z}, t)/\partial t^k|_{t=0}$, which is stable by Lemma 2.1 (c, g). $\square$

Polynomials $g, h \in \mathbb{R}[\mathbf{z}]$ are in *proper position*, denoted by $g \ll h$, if the polynomial $h + ig$ is stable. This is the multivariate analogue of interlacing roots for univariate polynomials with only real roots.

Proposition 2.3 (Lemma 2.8 of [2] and Theorem 1.6 of [1])**.** *Let $g, h \in \mathbb{R}[\mathbf{z}]$.*
(a) *Then $h \ll g$ if and only if $g + th \in \mathbb{R}[\mathbf{z}, t]$ is stable.*
(a) *Then $ag + bh$ is stable for all $a, b \in \mathbb{R}$ if and only if either $h \ll g$ or $g \ll h$.*

It then follows from Lemma 2.1(d,g) that if $h \ll g$ then both h and g are stable (or identically zero).

Proposition 2.4 (Lemma 2.6 of [1])**.** *Suppose that $g \in \mathbb{R}[\mathbf{z}]$ is stable. Then the sets*

$$\{h \in \mathbb{R}[\mathbf{z}] : g \ll h\} \quad and \quad \{h \in \mathbb{R}[\mathbf{z}] : h \ll g\}$$

are convex cones containing g.

Proposition 2.5. *Let V be a real vector space, $\phi : V^n \to \mathbb{R}$ a multilinear form, and $e_1, \ldots, e_n, v_2, \ldots, v_n$ fixed vectors in V. Suppose that the polynomial*

$$\phi(e_1, v_2 + z_2 e_2, \ldots, v_n + z_n e_n)$$

in $\mathbb{R}[\mathbf{z}]$ is not identically zero. Then the set of all $v_1 \in V$ for which the polynomial

$$\phi(v_1 + z_1 e_1, v_2 + z_2 e_2, \ldots, v_n + z_n e_n)$$

is stable is either empty or a convex cone (with apex 0) containing e_1 and $-e_1$.

Proof. Let C be the set of all $v_1 \in V$ for which the polynomial $\phi(v_1 + z_1 e_1, v_2 + z_2 e_2, \ldots, v_n + z_n e_n)$ is stable. For $v \in V$ let $F_v = \phi(v, v_2 + z_2 e_2, \ldots, v_n + z_n e_n)$. Since

$$\phi(v_1 + z_1 e_1, v_2 + z_2 e_2, \ldots, v_n + z_n e_n) = F_{v_1} + z_1 F_{e_1},$$

we have $C = \{v \in V : F_{e_1} \ll F_v\}$. Moreover since $F_{\lambda v + \mu w} = \lambda F_v + \mu F_w$ it follows from Proposition 2.4 that C is a convex cone provided that C is non-empty. If C is nonempty then $F_v + z_1 F_{e_1}$ is stable for some $v \in V$. But then F_{e_1} is stable, and so is

$$(\pm 1 + z_1) F_{e_1} = \phi(\pm e_1 + z_1 e_1, v_2 + z_2 e_2, \ldots, v_n + z_n e_n)$$

which proves that $\pm e_1 \in C$. $\square$

(Of course, by permuting the indices Proposition 2.5 applies to any index j as well.)

Let $\mathbb{C}[\mathbf{z}]^{\mathsf{ma}}$ denote the vector subspace of *multiaffine* polynomials: that is, polynomials of degree at most one in each indeterminate.

Proposition 2.6 (Theorem 5.6 of [4]). *Let $f \in \mathbb{R}[\mathbf{z}]^{\mathsf{ma}}$ be multiaffine. The following are equivalent:*
(a) *f is real stable.*
(b) *For all $1 \le i < j \le n$ and all $\mathbf{a} \in \mathbb{R}^n$, $f_i(\mathbf{a})f_j(\mathbf{a}) - f(\mathbf{a})f_{ij}(\mathbf{a}) \ge 0$, in which the subscripts denote partial differentiation.*

A linear transformation $T : \mathbb{C}[\mathbf{z}] \to \mathbb{C}[\mathbf{z}]$ *preserves stability* if $T(f)(\mathbf{z})$ is stable whenever $f(\mathbf{z})$ is stable.

Proposition 2.7 (Theorem 1.1 of [2]). *Let $T : \mathbb{C}[\mathbf{z}]^{\mathsf{ma}} \to \mathbb{C}[\mathbf{z}]$ be a linear transformation. Then T preserves stability if and only if either*
(a) *$T(f) = \eta(f) \cdot p$ for some linear functional $\eta : \mathbb{C}[\mathbf{z}]^{\mathsf{ma}} \to \mathbb{C}$ and stable $p \in \mathbb{C}[\mathbf{z}]$, or*
(b) *the polynomial $G_T(\mathbf{z}, \mathbf{w}) = T \prod_{j=1}^n (z_j + w_j)$ is stable in $\mathbb{C}[\mathbf{z}, \mathbf{w}]$.*

The *complexification* $T_{\mathbb{C}} : \mathbb{C}[\mathbf{z}] \to \mathbb{C}[\mathbf{z}]$ of a linear transformation $T : \mathbb{R}[\mathbf{z}] \to \mathbb{R}[\mathbf{z}]$ is defined as follows. For any $f \in \mathbb{C}[\mathbf{z}]$ write $f = g + \mathrm{i}h$ with $g, h \in \mathbb{R}[\mathbf{z}]$, which can be done uniquely. Then $T_{\mathbb{C}}(f) = T(g) + \mathrm{i}T(h)$. Let also $\hat{T}_{\mathbb{C}} = T(g) - \mathrm{i}T(h)$.

Proposition 2.8. *Let $T : \mathbb{R}[\mathbf{z}] \to \mathbb{R}[\mathbf{z}]$ be a linear transformation and let $f \in \mathbb{C}|z]$ be a stable polynomial. If T preserves real stability then either $T_{\mathbb{C}}(f)$ is stable, or $\hat{T}_{\mathbb{C}}(f)$ is stable.*

Proof. Let $f = g + \mathrm{i}h \in \mathbb{C}[\mathbf{z}]$ with $g, h \in \mathbb{R}[\mathbf{z}]$, and assume that f is stable. Then $h \ll g$ by definition. By Proposition 2.3(b), it follows that $ah + bg$ is real stable for all $a, b \in \mathbb{R}$. Therefore, $aT(h) + bT(g)$ is real stable for all $a, b \in \mathbb{R}$. By Proposition 2.3(b) again, either $T(h) \ll T(g)$ or $T(g) \ll T(h)$. By Proposition 2.3(a), either $T(g) + \mathrm{i}T(h) = T_{\mathbb{C}}(f)$ is stable, or $T(g) - \mathrm{i}T(h) = \hat{T}_{\mathbb{C}}(f)$ is stable. $\qquad \square$

3. Proof of the MMCPC.

3.1. Reduction to Ferrers matrices.

Lemma 3.1. *If $\operatorname{per}(z_j + a_{ij})$ is stable for all Ferrers matrices, then the MMCPC is true.*

Proof. If $\operatorname{per}(z_j + a_{ij})$ is stable for all Ferrers matrices, then by permuting the columns of such a matrix, the same is true for all monotone column $\{0, 1\}$-matrices. Now let $A = (a_{ij})$ be an arbitrary n-by-n monotone column matrix. We will show that $\operatorname{per}(z_j + a_{ij})$ is stable by n applications of Proposition 2.5.

Let V be the vector space of column vectors of length n. The multilinear form ϕ we consider is the permanent of an n-by-n matrix obtained by concatenating n vectors in V. Let each of $e_1, \ldots, e_n$ be the all-ones vector in V.

Initially, let $v_1, v_2, \ldots, v_n$ be arbitrary monotone $\{0, 1\}$-vectors. Then $\phi(v_1 + z_1e_1, \ldots, v_n + z_ne_n) = \operatorname{per}(J_nZ_n + H)$ for some monotone column $\{0, 1\}$-matrix H. One can specialize any number of v_j to the zero vector, and any number of z_j to 1, and the result is not identically zero. By hypothesis, all these polynomials are stable.

Now we proceed by induction. Assume that if $v_1, \ldots, v_{j-1}$ are the first $j-1$ columns of A, and if $v_j, \ldots, v_n$ are arbitrary monotone $\{0,1\}$-columns, then $\phi(v_1 + z_1 e_1, \ldots, v_n + z_n e_n)$ is stable. (The base case, $j = 1$, is the previous paragraph.) Putting $v_j = 0$ and $z_j = 1$, the resulting polynomial is not identically zero. By Proposition 2.5 (applied to index j), the set of vectors v_j such that $\phi(v_1 + z_1 e_1, \ldots, v_n + z_n e_n)$ is stable is a convex cone containing $\pm e_j$. Moreover, it contains all monotone $\{0,1\}$-columns, by hypothesis. Now, any monotone column of real numbers can be written as a nonnegative linear combination of $-e_1$ and monotone $\{0,1\}$-columns, and hence is in this cone. Thus, we may take $v_1, \ldots, v_{j-1}, v_j$ to be the first j columns of A, $v_{j+1}, \ldots, v_n$ to be arbitrary monotone $\{0,1\}$-columns, and the resulting polynomial is stable. This completes the induction step.

After the n-th step we find that $\mathrm{per}(J_n Z_n + A)$ is stable. $\qquad\square$

3.2. **A more symmetrical problem.** Let $A = (a_{ij})$ be an n-by-n Ferrers matrix, and let $\mathbf{z} = \{z_1, \ldots, z_n\}$. For each $1 \leq j \leq n$, let $y_j = (z_j + 1)/z_j$, and let $Y_n = \mathrm{diag}(y_1, \ldots, y_n)$. The matrix obtained from $J_n Z_n + A$ by factoring z_j out of column j for all $1 \leq j \leq n$ is $A Y_n + J_n - A = (a_{ij} y_j + 1 - a_{ij})$. It follows that

$$\mathrm{per}(z_j + a_{ij}) = z_1 z_2 \cdots z_n \cdot \mathrm{per}(a_{ij} y_j + 1 - a_{ij}). \qquad (3.1)$$

Lemma 3.2. *For a Ferrers matrix $A = (a_{ij})$, $\mathrm{per}(z_j + a_{ij})$ is stable if and only if $\mathrm{per}(a_{ij} y_j + 1 - a_{ij})$ is stable.*

Proof. The polynomials are not identically zero. Notice that $\mathrm{Im}(z_j) > 0$ if and only if $\mathrm{Im}(y_j) = \mathrm{Im}(1 + z_j^{-1}) < 0$. If $\mathrm{per}(z_j + a_{ij})$ is stable, then $\mathrm{per}(a_{ij} y_j + 1 - a_{ij}) \neq 0$ whenever $\mathrm{Im}(y_j) < 0$ for all $1 \leq j \leq n$. Since this polynomial has real coefficients, it follows that it is stable. The converse is similar. $\qquad\square$

The set of n-by-n Ferrers matrices has the following duality $A \mapsto A^\vee = J_n - A^\top$: transpose and exchange zeros and ones. That is, $A^\vee = (a_{ij}^\vee)$ in which $a_{ij}^\vee = 1 - a_{ji}$ for all $1 \leq i, j \leq n$. Note that $(A^\vee)^\vee = A$. However, the form of the expression $\mathrm{per}(a_{ij} y_j + 1 - a_{ij})$ is not preserved by this symmetry. To remedy this defect, introduce new indeterminates $\mathbf{x} = \{x_1, \ldots, x_n\}$ and consider the matrix $B(A) = (b_{ij})$ with entries $b_{ij} = a_{ij} y_j + (1 - a_{ij}) x_i$ for all $1 \leq i, j \leq n$. For example, if

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{then} \quad B(A) = \begin{bmatrix} x_1 & y_2 & y_3 & y_4 & y_5 \\ x_2 & x_2 & y_3 & y_4 & y_5 \\ x_3 & x_3 & y_3 & y_4 & y_5 \\ x_4 & x_4 & x_4 & y_4 & y_5 \\ x_5 & x_5 & x_5 & x_5 & x_5 \end{bmatrix}.$$

For emphasis, we may write $B(A; \mathbf{x}; \mathbf{y})$ to indicate that the row variables are $\mathbf{x}$ and the column variables are $\mathbf{y}$. The matrices $B(A)$ and $B(A^\vee)$ have the same general form, and in fact

$$\mathrm{per}(B(A^\vee; \mathbf{x}; \mathbf{y})) = \mathrm{per}(B(A; \mathbf{y}; \mathbf{x})). \qquad (3.2)$$

Clearly $\mathrm{per}(B(A))$ specializes to $\mathrm{per}(a_{ij} y_j + 1 - a_{ij})$ by setting $x_i = 1$ for all $1 \leq i \leq n$. We will show that $\mathrm{per}(B(A))$ is stable, for any Ferrers matrix A. By Lemmas 2.1(d), 3.2, and 3.1, this will imply the MMCPC.

$$\begin{bmatrix} . & . & \square & . & . & . \\ \square & . & . & . & . & . \\ . & . & . & . & . & \square \\ . & . & . & . & \square & . \\ . & \square & . & . & . & . \\ . & . & . & \square & . & . \end{bmatrix} \mapsto \begin{bmatrix} . & . & \square & . & . \\ \square & . & . & . & . \\ . & . & . & \square & . \\ . & . & . & . & \square \\ . & \square & . & . & . \end{bmatrix}$$

FIGURE 1. $\sigma = 3\ 1\ 6\ 5\ 2\ 4$ maps to $\pi(\sigma) = 3\ 1\ 4\ 5\ 2$.

3.3. **A differential recurrence relation.** Next, we derive a differential recurrence relation for polynomials of the form $\mathrm{per}(B(A))$, for A an n-by-n Ferrers matrix. There are two cases: either $a_{nn} = 0$ or $a_{nn} = 1$. Replacing A by $A^\vee$ and using (3.2), if necessary, we can assume that $a_{nn} = 0$.

Lemma 3.3. *Let $A = (a_{ij})$ be an n-by-n Ferrers matrix with $a_{nn} = 0$, let $k \geq 1$ be the number of 0's in the last column of A, and let A° be the matrix obtained from A by deleting the last column and the last row of A. Then*

$$\mathrm{per}(B(A)) = kx_n\,\mathrm{per}(B(A^\circ)) + x_ny_n\,\partial\mathrm{per}(B(A^\circ)),$$

in which

$$\partial = \sum_{i=1}^{n-k} \frac{\partial}{\partial x_i} + \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j}.$$

Proof. In the permutation expansion of $\mathrm{per}(B(A))$ there are two types of terms: those that do not contain y_n and those that do. Let T_σ be the term of $\mathrm{per}(B(A))$ indexed by $\sigma \in \mathfrak{S}_n$. For each $n - k + 1 \leq i \leq n$, let $\mathcal{C}_i$ be the set of those terms T_σ such that $\sigma(i) = n$; for a term in $\mathcal{C}_i$ the variable chosen in the last column is x_i. Let $\mathcal{D}$ be the set of all other terms; for a term in $\mathcal{D}$ the variable chosen in the last column is y_n.

For every permutation $\sigma \in \mathfrak{S}_n$, let (i_σ, j_σ) be such that $\sigma(i_\sigma) = n$ and $\sigma(n) = j_\sigma$, and define $\pi(\sigma) \in \mathfrak{S}_{n-1}$ by putting $\pi(i) = \sigma(i)$ if $i \neq i_\sigma$, and $\pi(i_\sigma) = j_\sigma$ (if $i_\sigma \neq n$). Let $T_{\pi(\sigma)}$ be the corresponding term of $\mathrm{per}(B(A^\circ))$. See Figure 1 for an example. Informally, $\pi(\sigma)$ is obtained from σ, in word notation, by replacing the largest element with the last element, unless the largest element is last, in which case it is deleted.

For each $n - k + 1 \leq i \leq n$, consider all permutations σ indexing terms in $\mathcal{C}_i$. The mapping $T_\sigma \mapsto T_{\pi(\sigma)}$ is a bijection from the terms in $\mathcal{C}_i$ to all the terms in $\mathrm{per}(B(A^\circ))$. Also, for each $\sigma \in \mathcal{C}_i$, $T_\sigma = x_n T_{\pi(\sigma)}$. Thus, for each $n - k + 1 \leq i \leq n$, the sum of all terms in $\mathcal{C}_i$ is $x_n\mathrm{per}(B(A^\circ))$.

Next, consider all permutations σ indexing terms in $\mathcal{D}$. The mapping $T_\sigma \mapsto T_{\pi(\sigma)}$ is $(n - k)$-to-one from $\mathcal{D}$ to the set of all terms in $\mathrm{per}(B(A^\circ))$, since one needs both $\pi(\sigma)$ and i_σ to recover σ. Let v_σ be the variable in position (i_σ, j_σ) of $B(A^\circ)$. Then $v_\sigma T_\sigma = x_n y_n T_{\pi(\sigma)}$. It follows that for any variable w in the set $\{x_1, \ldots, x_{n-k}, y_1, \ldots, y_{n-1}\}$, the sum over all terms in $\mathcal{D}$ such that $v_\sigma = w$ is

$$x_n y_n \frac{\partial}{\partial w}\mathrm{per}(B(A^\circ)).$$

Since v_σ is any element of the set $\{x_1, \ldots, x_{n-k}, y_1, \ldots, y_{n-1}\}$, it follows that the sum of all terms in $\mathcal{D}$ is $x_n y_n \partial\mathrm{per}(B(A^\circ))$.

The preceding paragraphs imply the stated formula. □

3.4. **Finally, proof of the MMCPC.**

Theorem 3.4. *For any n-by-n Ferrers matrix A, $\operatorname{per}(B(A))$ is stable.*

Proof. As above, by replacing A by $A^\vee$ if necessary, we may assume that $a_{1n} = 0$. We proceed by induction on n, the base case $n = 1$ being trivial. For the induction step, let A° be as in Lemma 3.3. By induction, we may assume that $\operatorname{per}(B(A^\circ))$ is stable; clearly this polynomial is multiaffine. Thus, by Lemma 3.3, it suffices to prove that the linear transformation $T = k + y_n \partial$ maps stable multiaffine polynomials to stable polynomials if $k \geq 1$. This operator has the form $T = k + z_m \sum_{j=1}^{m-1} \partial/\partial z_j$ (renaming the variables suitably). By Proposition 2.7 it suffices to check that the polynomial

$$
\begin{aligned}
G_T(\mathbf{z}, \mathbf{w}) &= T \prod_{j=1}^{m} (z_j + w_j) \\
&= \left(k + z_m \sum_{j=1}^{m-1} \frac{1}{z_j + w_j} \right) \prod_{j=1}^{m} (z_j + w_j)
\end{aligned}
$$

is stable. If z_j and w_j have positive imaginary parts for all $1 \leq j \leq m$ then

$$
\xi = \frac{k}{z_m} + \sum_{j=1}^{m-1} \frac{1}{z_j + w_j}
$$

has negative imaginary part (since $k \geq 0$). Thus $z_m \xi \neq 0$. Also, $z_j + w_j$ has positive imaginary part, so that $z_j + w_j \neq 0$ for each $1 \leq j \leq m$. It follows that $G_T(\mathbf{z}, \mathbf{w}) \neq 0$, so that G_T is stable, completing the induction step and the proof. □

Proof of the MMCPC. Let A be any n-by-n Ferrers matrix. By Theorem 3.4, $\operatorname{per}(B(A))$ is stable. Specializing $x_i = 1$ for all $1 \leq i \leq n$, Lemma 2.1(d) implies that $\operatorname{per}(a_{ij} y_j + 1 - a_{ij})$ is stable. Now Lemma 3.2 implies that $\operatorname{per}(z_j + a_{ij})$ is stable. Finally, Lemma 3.1 implies that the MMCPC is true. □

Henceforth, we shall refer to the MMCPT – "T" for "Theorem".

4. FURTHER RESULTS.

4.1. **Generalization to rectangular matrices.** We can generalize Theorem 3.4 to rectangular matrices, as follows. Let $A = (a_{ij})$ be an m-by-n matrix. As in the square case, A is a *Ferrers matrix* if it is a $\{0, 1\}$-matrix that is weakly decreasing down columns and increasing across rows. The matrix $B(A)$ is constructed just as in the square case: $B(A) = (b_{ij}) = (a_{ij} y_j + (1 - a_{ij}) x_i)$, using row variables $\mathbf{x} = \{x_1, \ldots, x_m\}$ and column variables $\mathbf{y} = \{y_1, \ldots, y_n\}$. For emphasis, we may write $B(A; \mathbf{x}; \mathbf{y})$ to indicate that the row variables are $\mathbf{x}$ and the column variables are $\mathbf{y}$. The symmetry $A \mapsto A^\vee$ takes an m-by-n Ferrers matrix to an n-by-m Ferrers matrix.

Now let $k \leq \min\{m, n\}$. The *k-permanent* of an m-by-n matrix $H = (h_{ij})$ is

$$
\operatorname{per}_k(H) = \sum_R \sum_C \sum_\beta \prod_{i \in R} h_{i, \beta(i)},
$$

in which R ranges over all k-element subsets of $\{1, \ldots, m\}$, C ranges over all k-element subsets of $\{1, \ldots, n\}$, and β ranges over all bijections from R to C. (Note that $\mathrm{per}_0(H) = 1$ for any matrix H.) In the case $k = m = n$ this reduces to the permanent of a square matrix.

For an m-by-n Ferrers matrix A and $k \leq \min\{m, n\}$, note that

$$\mathrm{per}_k(B(A^\vee; \mathbf{x}; \mathbf{y})) = \mathrm{per}_k(B(A; \mathbf{y}; \mathbf{x})). \tag{4.1}$$

Thus, replacing A by $A^\vee$ if necessary, we may assume that $m \leq n$.

Proposition 4.1. *Let* $A = (a_{ij})$ *be an* m-by-n *Ferrers matrix. Then* $\mathrm{per}_k(B(A))$ *is stable for all* $k \leq \min\{m, n\}$.

Proof. Using (4.1), if necessary, we may assume that $m \leq n$.

We begin by showing that $\mathrm{per}_m(B(A))$ is stable. Let A' be the n-by-n Ferrers matrix obtained by concatenating $n - m$ rows of 0's to the bottom of A. One checks that

$$\mathrm{per}(B(A')) = (n - m)! \, x_n x_{n-1} \cdots x_{m+1} \cdot \mathrm{per}_m(B(A)).$$

By Theorem 3.4, $\mathrm{per}(B(A'))$ is stable, and it follows easily that $\mathrm{per}_m(B(A))$ is stable.

Now, let $J_{m,n}$ be the m-by-n matrix of all 1's. Then

$$\mathrm{per}_m(B(A) + t J_{m,n}) = \sum_{k=0}^{m} \mathrm{per}_k(B(A)) \binom{n-k}{m-k} (m-k)! t^{m-k}.$$

By Lemma 2.1(c,f), this polynomial is stable. Extracting the coefficient of t^{m-k} from this, and dividing by $\binom{n-k}{m-k}(m-k)!$, Lemma 2.2 shows that $\mathrm{per}_k(B(A))$ is stable for all $0 \leq k \leq m$. $\qquad\square$

Proposition 4.1 suggests the idea of a similar generalization of the MMCPT: is it true that $\mathrm{per}_k(J_{m,n} Z_n + A)$ is stable for every m-by-n monotone column matrix A and $k \leq \min\{m, n\}$? This conjecture originates in [8]. One cannot derive this from Proposition 4.1, however, because there is no analogue of (3.1) for k-permanents. Nonetheless, we can prove this result for half the cases.

Proposition 4.2. *Let* A *be an* m-by-n *monotone column matrix with* $m \geq n$, *and let* $k \leq n$. *Then* $\mathrm{per}_k(J_{m,n} Z_n + A)$ *is stable.*

Proof. Let A' be the m-by-m matrix obtained from A by concatenating $m - n$ zero columns to the right of A. Then

$$\mathrm{per}(J_m Z_m + A') = (m - n)! \, z_m z_{m-1} \cdots z_{n+1} \cdot \mathrm{per}_n(J_{m,n} Z_n + A).$$

Since $\mathrm{per}(J_m Z_m + A')$ is stable, it follows that $\mathrm{per}_n(J_{m,n} Z_n + A)$ is stable. By Lemma 2.1(c,f), it follows that $\mathrm{per}_n(J_{m,n} Z_n + A + t J_{m,n})$ is stable. Extracting the coefficient of t^{n-k} from this, and dividing by $\binom{m-k}{n-k}(n-k)!$, Lemma 2.2 shows that $\mathrm{per}_k(J_{m,n} Z_n + A)$ is stable for all $0 \leq k \leq n$. $\qquad\square$

4.2. **A cycle-counting generalization.** Theorem 3.4 can be generalized in another direction, as follows.

For each permutation $\sigma \in \mathfrak{S}_n$, let $\mathrm{cyc}(\sigma)$ denote the number of cycles of σ. For an indeterminate α and an n-by-n matrix $H = (h_{ij})$, the α-*permanent* of H is

$$\mathrm{per}(H; \alpha) = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\mathrm{cyc}(\sigma)} \prod_{i=1}^{n} h_{i, \sigma(i)}.$$

The numbers $\text{cyc}(\sigma)$ behave well with respect to the duality $A \mapsto A^\vee$: for any Ferrers matrix A,

$$\text{per}(B(A^\vee; \mathbf{x}, \mathbf{y}); \alpha) = \text{per}(B(A; \mathbf{y}, \mathbf{x}); \alpha). \tag{4.2}$$

Lemma 4.3. *Let $A = (a_{ij})$ be an n-by-n Ferrers matrix with $a_{nn} = 0$, let $k \geq 1$ be the number of 0's in the last column of A, and let A° be the matrix obtained from A by deleting the last column and the last row of A. Then*

$$\text{per}(B(A); \alpha) = (\alpha + k - 1)x_n \, \text{per}(B(A^\circ); \alpha) + x_n y_n \, \partial\text{per}(B(A^\circ); \alpha),$$

with ∂ as in Lemma 3.3.

Proof. Adopt the notation of the proof of Lemma 3.3. To obtain the present result, observe that if T_σ is in $\mathcal{C}_n$ then $\text{cyc}(\sigma) = 1 + \text{cyc}(\pi(\sigma))$, and otherwise $\text{cyc}(\sigma) = \text{cyc}(\pi(\sigma))$. $\qquad\square$

Proposition 4.4. *For $\alpha > 0$ and A a Ferrers matrix, $\text{per}(B(A); \alpha)$ is stable.*

Proof. Reprising the proof of Theorem 3.4, it suffices to show that an operator of the form

$$T = (\alpha + k - 1) + z_m \sum_{i=1}^{m-1} \partial/\partial z_i$$

preserves stability when $k \geq 1$. The argument of the proof of Theorem 3.4 works when $\alpha > 0$. $\qquad\square$

For $\alpha > 0$ and A a Ferrers matrix, specialize all $x_i = 1$ and diagonalize all $y_j = z$ in $\text{per}(B(A); \alpha)$. By Lemma 2.1(c,d), the result is a (univariate) polynomial with only real roots. This special case is also implied by Theorem 2.5 of [6].

4.3. **Multivariate stable Eulerian polynomials.** Given a permutation $\sigma \in \mathfrak{S}_n$, viewed as a linear sequence $\sigma(1)\sigma(2)\cdots\sigma(n)$, let $\text{L}(\sigma)$ denote the result of the following procedure. First form the two-line array

$$\text{T}(\sigma) = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

Then, viewing $\text{T}(\sigma)$ as a map sending i to $\sigma(i)$, break $\text{T}(\sigma)$ into cycles, with the smallest element of each cycle at the end, and cycles listed left-to-right with smallest elements increasing. Finally, erase the parentheses delimiting the cycles to form a new linear sequence $\text{L}(\sigma)$. For example, if $\sigma = 341526978$, then $\text{T}(\sigma)$ has cycle decomposition $(31)(452)(6)(987)$ and $\text{L}(\sigma) = 314526987$.

Let $\text{P}(\sigma)$ denote the placement of n nonattacking rooks on the squares $(i, \sigma(i))$, $1 \leq i \leq n$. As noted by Riordan [10], rooks in $\text{P}(\sigma)$ that occur above the diagonal correspond to exceedences (values of i for which $\sigma(i) > i$) in σ and descents (values of i for which $\sigma(i) > \sigma(i+1)$) in $\text{L}(\sigma)$. Hence

$$\sum_{\sigma \in \mathfrak{S}_n} z^{\text{exc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} z^{\text{des}(\sigma)}, \tag{4.3}$$

where $\text{exc}(\sigma)$ is the number of exceedences and $\text{des}(\sigma)$ the number of descents of σ.

The polynomial in (4.3) is known as the *Eulerian polynomial*. It is one of the classic examples in combinatorics of a polynomial with only real roots. Let $E_n = (e_{ij})$ denote the n-by-n matrix with $e_{ij} = 0$ if $i \geq j$ and $e_{ij} = 1$ if $i < j$. Then $\text{per}(B(E_n; \mathbf{x}, \mathbf{y}))$ is stable by Theorem 3.4. Let $\mathbf{1}$ be the vector (of appropriate size) of all ones. From the above discussion, $\text{per}(B(E_n; \mathbf{1}; z, \ldots, z))$ equals the Eulerian

polynomial; by Lemma 2.1(c,d) this is a univariate stable polynomial with real coefficients, so it has only real roots. Similarly, we see that

$$\operatorname{per}(B(E_n; \mathbf{1}; \mathbf{y})) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{\sigma(i) > \sigma(i+1)} y_{\sigma(i)} \tag{4.4}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} \prod_{\sigma(i) > i} y_{\sigma(i)} \tag{4.5}$$

is stable in $\{y_1, y_2, \ldots, y_n\}$ (but y_1 does not really occur).

Letting $f = \operatorname{per}(B(E_n; \mathbf{1}; \mathbf{y}))$, note that the partial of f with respect to y_i, evaluated at all y-variables equal to 1, equals the number of permutations in $\mathfrak{S}_n$ that have i as a "descent top", i.e. have the property that i is followed immediately by something less than i. Denoting this number by $\operatorname{Top}(i; n)$, applying Proposition 2.6 to f we get

$$\operatorname{Top}(i; n)\operatorname{Top}(j; n) \geq n! \operatorname{Top}(i, j; n), \tag{4.6}$$

where $\operatorname{Top}(i, j; n)$ is the number of permutations in $\mathfrak{S}_n$ having both i and j as descent tops, with $2 \leq i < j \leq n$. Dividing both sides of the above equation by $n!^2$ shows that occurrences of descent tops in a uniformly random permutation are negatively correlated.

More general forms of (4.4) can be defined which still maintain stability. First of all, cycles in $\operatorname{T}(\sigma)$ clearly translate into left-to-right minima in $\operatorname{L}(\sigma)$, and so by Proposition 4.4 the polynomial

$$\sum_{\sigma \in \mathfrak{S}_n} \alpha^{\operatorname{LRmin}(\sigma)} \prod_{\sigma(i) > \sigma(i+1)} y_{\sigma(i)}$$

is stable in the $\mathbf{y}$ for $\alpha > 0$, with $\operatorname{LRmin}(\sigma)$ denoting the number of left-to-right minima of σ.

Secondly, the sum in (4.4) can be replaced by a sum over permutations of a multiset. For a given vector $\mathbf{v} = (v_1, \ldots, v_t) \in \mathbb{N}^t$, let $N(\mathbf{v}) = \{1^{v_1} 2^{v_2} \cdots t^{v_t}\}$ denote the multiset with v_i copies of i, and $\mathcal{M}(\mathbf{v})$ the set of multiset permutations of $N(\mathbf{v})$ (so for example $\mathcal{M}(2, 1) = \{112, 121, 211\}$). Riordan [10] noted that if we map our previous sequence $L(\sigma)$ to a multiset permutation $m(\sigma)$ by replacing numbers 1 through v_1 in $L(\sigma)$ by 1's, numbers $v_1 + 1$ through $v_1 + v_2$ by 2's, etc., we get a $\prod v_i!$ to 1 map, and furthermore certain squares above the diagonal where rooks in $P(\sigma)$ correspond to descents in $L(\sigma)$ no longer correspond to descents in $m(\sigma)$. For example, if $v_1 = 2$, then 1 and 2 in $L(\sigma)$ both get mapped to 1 in $m(\sigma)$, so a rook on square $(1, 2)$ no longer corresponds to a descent.

Let n denote the sum of the coordinates of $\mathbf{v}$, and let $Y(\mathbf{v})$ be the sequence of variables obtained by starting with $y_1, \ldots, y_n$ and setting the first v_1 y-variables equal to y_1, the next v_2 y-variables equal to y_2, etc. Then if $E(\mathbf{v})$ is the Ferrers matrix whose first v_1 columns are all zeros, the next v_2 columns have ones in the top v_1 rows and zeros below, the next v_3 columns have ones in the top $v_1 + v_2$ rows and zeros below, etc., an easy extension of the argument above implies

$$(1/ \prod_i v_i!)\operatorname{per}(B(E(\mathbf{v}; \mathbf{1}; Y(\mathbf{v})))) = \sum_{\sigma \in \mathcal{M}(\mathbf{v})} \prod_{\sigma(i) > \sigma(i+1)} y_{\sigma(i)} \tag{4.7}$$

is stable in the y_i. This contains Simion's result [11], that the multiset Eulerian polynomial has only real roots. If $\mathbf{v}$ has all ones, i.e. $N(\mathbf{v})$ is a set, it reduces to our previous result. Finally, we note that this argument also shows that if we replace

the condition $\sigma(i) > \sigma(i+1)$ by the more general condition $\sigma(i) > \sigma(i+1)+j-1$, for any fixed positive integer j, we still get stability.

4.4. Grace's Apolarity Theorem. Univariate complex polynomials $f(t) = \sum_{k=0}^{n} \binom{n}{k} a_k t^k$ and $g(t) = \sum_{k=0}^{n} \binom{n}{k} b_k t^k$ are *apolar* if $a_n b_n \neq 0$ and

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_k b_{n-k} = 0.$$

Lemma 4.5. *Let $f(t) = \sum_{k=0}^{n} \binom{n}{k} a_k t^k$ and $g(t) = \sum_{k=0}^{n} \binom{n}{k} b_k t^k$ be complex polynomials of degree n. Let the roots of $f(t)$ be $z_1, \ldots, z_n$ and let the roots of $g(t)$ be $w_1, \ldots, w_n$. Then*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_k b_{n-k} = n! \, a_n b_n \operatorname{per}(w_i - z_j).$$

In particular, $f(t)$ and $g(t)$ are apolar if and only if $\operatorname{per}(w_i - z_j) = 0$.

Proof. It suffices to prove this for monic polynomials $f(t)$ and $g(t)$. For each permutation $\sigma \in \mathfrak{S}_n$ there are 2^n terms in $\operatorname{per}(w_i - z_j)$, since for each $1 \leq i \leq n$ either w_i or $-z_{\sigma(i)}$ can be chosen. For each subset R of rows of size k, and subset C of columns of size $n-k$, the monomial $\mathbf{w}^R \mathbf{z}^C$ is produced $k!(n-k)!$ times. Since $(-1)^k a_{n-k}$ is the k-th elementary symmetric function of $\{z_1, \ldots, z_n\}$, and similarly for $(-1)^k b_{n-k}$ and $\{w_1, \ldots, w_n\}$, the result follows. $\square$

Lemma 4.6. *Let $f(t)$ and $g(t)$ be polynomials of degree n. Let $t \mapsto \phi(t) = (at+b)/(ct+d)$ be a Möbius transformation, with inverse $\phi^{-1}(t) = (\alpha t + \beta)/(\gamma t + \delta)$. Let $\widehat{f}(t) = (\gamma t + \delta)^n f(\phi^{-1}(t))$ and $\widehat{g}(t) = (\gamma t + \delta)^n g(\phi^{-1}(t))$ have degree n. Then $\widehat{f}(t)$ and $\widehat{g}(t)$ are apolar if and only if $f(t)$ and $g(t)$ are apolar.*

Proof. Let the roots of $f(t)$ be $z_1, \ldots, z_n$ and let the roots of $g(t)$ be $w_1, \ldots, w_n$. Then the roots of $\widehat{f}(t)$ are $\phi(z_1), \ldots, \phi(z_n)$ and the roots of $\widehat{g}(t)$ are $\phi(w_1), \ldots, \phi(w_n)$. Consider the permanent $\operatorname{per}(\phi(w_i) - \phi(z_j))$. The (i,j)-entry of this matrix is

$$\frac{aw_i + b}{cw_i + d} - \frac{az_j + b}{cz_j + d} = \frac{(ad-bc)(w_i - z_j)}{(cw_i + d)(cz_j + d)}.$$

Factor $(cw_i + d)^{-1}$ out of row i, factor $(cz_j + d)^{-1}$ out of column j, and factor $ad - bc$ out of every row. Therefore

$$\operatorname{per}(\phi(w_i) - \phi(z_j)) = \frac{(ad-bc)^n}{\prod_{h=1}^{n}(cw_h + d)(cz_h + d)} \cdot \operatorname{per}(w_i - z_j).$$

Since the prefactor on the right-hand side is neither zero nor infinite, the result follows from Lemma 4.5. $\square$

A *circular region* $\mathcal{A}$ is a proper subset of $\mathbb{C}$ that is either open or closed, and is bounded by either a circle or a straight line.

Theorem 4.7 (Grace's Apolarity Theorem). *Let $f(t)$ and $g(t)$ be apolar polynomials. If every root of $g(t)$ is in a circular region $\mathcal{A}$, then $f(t)$ has at least one root in $\mathcal{A}$.*

Proof. It suffices to prove this for monic polynomials $f(t)$ and $g(t)$, and for open circular regions $\mathcal{A}$ since a closed circular region is the intersection of all open circular regions which contain it. Let $t \mapsto \phi(t)$ be a Möbius transformation that maps $\mathcal{A}$ to the upper half-plane $\mathcal{H}$. By Lemma 4.6, it suffices to prove this when the circular region is $\mathcal{H}$ itself. Let the roots of $f(t)$ be $z_1, \ldots, z_n$ and let the roots of $g(t)$ be $w_1, \ldots, w_n$.

If all of $w_1, \ldots, w_n$ are real numbers, then by permuting the rows of $(w_i + z_j)$ we can assume that $w_1 \leq \cdots \leq w_n$ without changing the value of $\mathrm{per}(w_i + z_j)$. By the MMCPT, $\mathrm{per}(w_i + z_j)$ is a real stable polynomial in $z_1, \ldots, z_n$. In other words, the transformation $T : \mathbb{R}[z] \to \mathbb{R}[\mathbf{z}]$ defined by

$$T(f(z)) = \mathrm{per}(w_i + z_j),$$

where $f(z) = (z + w_1)(z + w_2) \cdots (z + w_n)$ preserves real stability. This is a linear transformation. Suppose that $f(z) \in \mathbb{C}[z]$ is stable. By Proposition 2.8, either $T_{\mathbb{C}}(f(z))$ is stable or $\hat{T}_{\mathbb{C}}(f(z))$ is stable. Diagonalizing by setting $z_j = z$ for all $1 \leq j \leq n$, we see that $T_{\mathbb{C}}(f)(z, \ldots, z) = n! f(z)$, so that $T_{\mathbb{C}}(f(z))$ is stable. Therefore $\mathrm{per}(w_i + z_j)$ is a stable polynomial in $\mathbb{C}[\mathbf{z}]$, for all $w_1, \ldots, w_n \in \mathcal{H}$. Therefore $\mathrm{per}(w_i + z_j)$ is a stable polynomial in $\mathbb{C}[\mathbf{w}, \mathbf{z}]$. Actually it satisfies a stronger stability property. Namely if $z_j \in \overline{\mathcal{H}}$ and $w_j \in \mathcal{H}$ for all $1 \leq j \leq n$, then $\mathrm{per}(w_i + z_j) \neq 0$. Indeed if we fix $\zeta_j \in \overline{\mathcal{H}}$ for all $1 \leq j \leq n$, then the polynomial $\mathrm{per}(w_i + \zeta_j) \in \mathbb{C}[\mathbf{w}]$ is either identically zero or stable (and not identically zero) by Lemma 2.1 (d). Clearly $\mathrm{per}(w_i + \zeta_j)$ is not identically zero.

Now assume that $w_i \in \mathcal{H}$ for all $1 \leq i \leq n$. Arguing for a contradiction, assume that $z_j \notin \mathcal{H}$ for all $1 \leq j \leq n$. Then $-z_j \in \overline{\mathcal{H}}$ for all $1 \leq j \leq n$. Hence $\mathrm{per}(w_i - z_j) \neq 0$ by the argument given in the previous paragraph. But $\mathrm{per}(w_i - z_j) = 0$ by Lemma 4.5, since $f(t)$ and $g(t)$ are apolar. This contradiction completes the proof. $\qquad\square$

Our proof of Grace's apolarity theorem relies on MMCPT. To avoid going in circles we should ensure ourselves that the proof of MMCPT does not use any form of Grace's apolarity theorem. In fact what we use in the proof of MMCPT is that condition (b) in Proposition 2.7 implies that the operator preserves stability, the proof of which does not use Grace's apolarity theorem.

4.5. **Permanental inequalities.** If A is a n-by-n matrix and $S \subseteq [n] = \{1, \ldots, n\}$ let A_S be the matrix obtained by replacing the columns indexed by S by columns of all ones.

Corollary 4.8. *Let A be an n-by-n monotone matrix, $S \subset [n]$, and $i, j \in [n] \setminus S$. Then*

$$\mathrm{per}(A_{S\cup\{i\}})\mathrm{per}(A_{S\cup\{j\}}) \geq \mathrm{per}(A_{S\cup\{i,j\}})\mathrm{per}(A_S).$$

Proof. The proof follows immediately from Proposition 2.6 applied to $\mathrm{per}(z_j + a_{ij})$. $\qquad\square$

Note that Corollary 4.8 can be interpreted as that the permanent is pairwise negatively associated in columns.

The *generating polynomial* of a discrete measure $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is given by

$$G_\mu(\mathbf{z}) = \sum_{S \in 2^{[n]}} \mu(S) \prod_{j \in S} z_j.$$

The measure μ is *Rayleigh* if

$$\frac{\partial G_\mu}{\partial z_i}(\mathbf{x})\frac{\partial G_\mu}{\partial z_j}(\mathbf{x}) \geq \frac{\partial^2 G_\mu}{\partial z_i \partial z_j}(\mathbf{x})G_\mu(\mathbf{x}), \tag{4.8}$$

for all $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$, and it is called *strongly Rayleigh* if (4.8) holds for all $\mathbf{x} \in \mathbb{R}^n$. We refer to [3, 12] for more information on Rayleigh and strongly Rayleigh measures.

Suppose that A is an n-by-n monotone matrix with nonnegative coefficients. Consider the discrete measure $\mu_A : 2^{[n]} \to \mathbb{R}_{\geq 0}$ defined by $\mu_A(S) = \text{per}(A_S)$. By Proposition 2.6 we see that μ_A is strongly Rayleigh. This fact entails many inequalities.

Corollary 4.9. *Suppose that A is an n-by-n monotone matrix with nonnegative coefficients. Then*

$$\text{per}(A_S)\text{per}(A_T) \geq \text{per}(A_{S\cup T})\text{per}(A_{S\cap T}),$$

for all $S, T \subseteq [n]$.
Moreover

$$\text{per}(A) \leq s_1 \cdots s_n \frac{n!}{n^n},$$

where s_i is the sum of the elements in the ith column.

Proof. The first inequality holds for all Rayleigh measures, see [12, Theorem 4.4].

Let $\mu(S) = \text{per}(A_{[n]\setminus S})/n!$. By the above $\mu(S)\mu(T) \geq \mu(S \cup T)\mu(S \cap T)$ for all $S, T \subseteq [n]$. Thus $\mu(S \cup T) \leq \mu(S)\mu(T)$ whenever $S \cap T = \emptyset$, and after iteration $\mu([n]) \leq \mu(\{1\}) \cdots \mu(\{n\})$. The proof now follows by observing that $\mu(\{i\}) = s_i/n$ for all $i \in [n]$.

One can also prove the last inequality by an elementary argument. If there are two different consecutive elements $b > a$ in a column of A, replace these by their average to obtain the matrix A'. It is plain to see that $\text{per}(A) \leq \text{per}(A')$. Iterating this procedure it follows that $\text{per}(A) \leq \text{per}(B)$, where each element in column i of B is equal to s_i/n. The inequality now follows. $\square$

The second inequality in Corollary 4.9 can be compared with the Van der Waerden conjecture which asserts that the permanent of a doubly stochastic matrix is greater than or equal to $n!/n^n$. The Van der Waerden conjecture which was stated in 1926 was proved by Falikman in 1981; the case of equality was proved by Egorychev. Gurvits has recently provided a beautiful proof of a vast generalization of the Van der Waerden conjecture using stable polynomials, see [5] and [13, Section 8].

If $f : 2^{[n]} \to \mathbb{R}$, and μ is a discrete measure on $2^{[n]}$ we let

$$\int f d\mu = \sum_{S \in 2^{[n]}} f(S)\mu(S).$$

A measure μ on $2^{[n]}$ is *negatively associated* if for all increasing functions $f, g : 2^{[n]} \to \mathbb{R}$ depending on disjoint sets of variables

$$\int fg d\mu \int d\mu \leq \int f d\mu \int g d\mu,$$

see e.g. [3].

Corollary 4.10. *Suppose that A is an n-by-n monotone matrix with nonnegative coefficients. Then the discrete measure $\mu_A : 2^{[n]} \to \mathbb{R}_{\geq 0}$, defined by $\mu_A(S) = \mathrm{per}(A_S)$, is negatively associated.*

Proof. The corollary follows from the fact that μ_A is strongly Rayleigh. Such measures are negatively associated, which was proved in [3, Theorem 4.9]   □

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